ADVANCES IN MATHEMATICAL MODELLING OF A SUBMARINE

Sadko Mandžuka

Control Engineering Department, Brodarski Institute, d.o.o., Av. V.Holjevca 20, 10000 Zagreb, CROATIA (E-mail: sadko@brod.hrbi.hr).

Abstract: A new approach for the mathematical modelling of submarine dynamics at the periscope depth is presented. In the papers published so far wave disturbances are modelled by a simple affine form. The main disadvantage of such an approach is a poor correspondence with the real nature of submarine dynamics at the periscope depth. Using the new modelling approach this disadvantage is avoided. The results of the computer simulation are presented.

Keywords: Mathematical models, Stochastic modelling, Phase correction

1. INTRODUCTION

The dynamic characteristics of many physical systems can be described by mathematical equations. It must be recognised that a model is an approximation which characterises the dynamic behaviour of a physical system. As it is an approximation, the degree to which the model is accurate varies. Generally speaking, the more accurate the model, the more complex it is. Thus, a key consideration in mathematical modelling is a trade-off between accuracy and complexity. Creating a more accurate and complex model requires also more effort and, therefore, more cost as well. In the case of more complex models the analysis costs are also increased because the designer must utilize more complex analysis tools (Sinha and Kuszta, 1983). Thus, a key consideration in mathematical modelling is a trade-off between accuracy and complexity.

As a submarine approaches the free surface, several complexities are introduced into the mathematical modelling approach (Musker, et al., 1988; Mandžuka, 1998a; Feng et al., 1996; Tolliver, 1996). The standard approach is first to establish a dynamic model appropriate for a deeply submerged submarine at moderate speeds. The forces and moments resulting from the seaway will then be superimposed on that model to provide a reasonable approximation to the submarine motion below waves. The low-frequency state space mathematical model of the submarine is given in Section 2.

The seaway disturbances are described in the mathematical form using different kinds of spectra. The most commonly used spectrum in ship motions studies is the Pierson-Moskowitz spectrum for fully developed seas (Lloyd, 1989; Fossen, 1994; Tabain, 1997). Unfortunately, the equations arising from such descriptions are difficult to manipulate in analytical investigations. Following the linear theory, using spectral factorisation techniques, the sea spectrum was modelled by transfer function representation (colour filter). The assumption which is fundamental for the development of behavioural models of a submarine at the periscope depth is that the sea state is known and can be described by a spectral density function (Fossen, 1994; Mandžuka, 1994). In the published submarine periscope depth control research, these disturbances were modelled by the simple affine form. The main disadvantage of this approach is a poor correspondence with the real situation. The dynamic model is inadequate, and there is no phase information between disturbance components. The new high-frequency mathematical model of the submarine dynamics is described in Section 3. The compact state space model of submarine at the periscope depth is given in Section 4. The numerical examples given in this paper are based on the mathematical model of the new Croatian Navy midget submarine VELEBIT. The submarine
represents an upgraded variant of one of the well-known midget submarines of the Una class (Musteric, 1996). The hydroplane system consists of two pairs of stern rudders (X-rudder configuration) and one pair of bow hydroplanes. A virtual stern hydroplane is considered in this paper.

2. LOW-FREQUENCY MODEL

The submarine dynamics in deep water, including the control forces and moments produced by the rudder and planes, are typically described by a set of six-degree-of-freedom nonlinear differential equations, based on the Kelvin-Kirchhoff hydrodynamical equations. In the past period, various simplified models have been developed and used in the study and control of the motion of a submarine and other underwater vehicles (Tolliver, 1996; Fossen, 1994; Gueler, 1989). The vertical motions of the submarine are represented by a set of the nonlinear state equations referring to the coordinate axes and notations given in Fig. 1.

Fig. 1. The submarine mathematical model notation

Assuming that speed U is constant, the equations of motions in the x-z plane can be written as:

\[
\begin{align*}
\dot{w}(t) &= \frac{ZU}{Lm_3} w(t) + \frac{1}{m_3} (Z' + m^2 U) \dot{q}(t) + \\
&+ \frac{Z_{\text{wave}} U^2}{Lm_3} \delta B(t) + \frac{Z_{\text{wave}} U^2}{Lm_3} \delta S(t) + \frac{2}{\rho Lm_3} Z_{\text{wave}}(t), \\
\dot{q}(t) &= \frac{M_{\text{wave}} U^2}{L_1^2} w(t) + \frac{M_{\text{wave}} U^2}{L_1^2} q(t) + \\
&+ \frac{M_{\text{wave}} U^2}{L_1^2} \delta B(t) + \frac{M_{\text{wave}} U^2}{L_1^2} \delta S(t) + \frac{2mg \zeta_{GB}}{\rho L_1^2 \theta(t)} + \frac{2}{\rho L_1^2} M_{\text{wave}}(t), \\
\dot{\theta}(t) &= \omega(t) \cos(\theta(t) - U \sin(\theta(t) \\
\theta(t) &= \dot{\theta}(t) = \dot{\theta}(t)
\end{align*}
\]

All primed quantities are non-dimensional and:

- m - submarine mass,
- \( \rho \) - sea water mass density,
- L - submarine length,
- \( I_y \) - moment of inertia about y-axis,
- \( \delta B(t), \delta S(t) \) - instantaneous bow and stern hydroplane deflections respectively,
- \( \zeta_{GB} \) - metacentric height,
- \( Z', M' \) - hydrodynamic coefficients,
- \( Z_{\text{wave}}(t), M_{\text{wave}}(t) \) - instantaneous force in heave and moment in pitch induced by waves.

The linearised submarine dynamics can be presented in the matrix form as:

\[
\begin{align*}
\dot{x}(t) &= A_x x(t) + B_x u(t) + G f(t) + H s(t) \\
x(t) &= [w(t), q(t), h(t), \theta(t)]^T \\
u(t) &= [\delta B(t), \delta S(t)]^T \\
f(t) &= [f_w, f_q]^T \\
s(t) &= [s_s, s_q]^T
\end{align*}
\]

where the system matrices are in the form:

\[
A_x = \begin{bmatrix} a_{11}U & a_{12}U & 0 & 0 \\
0 & a_{22}U & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{bmatrix}, \quad B_x = \begin{bmatrix} b_{11} \\
b_{12} \\
0 \\
0 \end{bmatrix}
\]

\[
G_v = \begin{bmatrix} g_{11} & 0 \\
0 & g_{22} \\
0 & 0 \\
0 & 0 \end{bmatrix}, \quad H_v = \begin{bmatrix} h_{11} \\
h_{12} \\
0 \\
0 \end{bmatrix}
\]

The instantaneous force in heave \( (Z_{\text{wave}}(t)) \), and moment in pitch \( (M_{\text{wave}}(t)) \), induced by waves, are included in the mathematical model as:

- oscillatory normed components \( (f_w, f_q) \),
- drift normed components \( (s_s', s_q') \).

The terms in matrices \( G_v \) and \( H_v \) will be determined in the next chapter.

3. HIGH-FREQUENCY MODEL

A submarine, when operating near the surface, is influenced by sea wave forces and moments. There are different regimes of interaction between a submerged body and a wave field. Broadly, they can be broken into several areas. Inertial interaction, where the body acts like a particle in the wave field. Wave diffraction, where the body's influence upon the wave field is accounted for. Finally, there are flow separation (viscous) effects. The relative importance of each of these effects can be determined by examining the relationship of the body size to the wave parameters, (Sarpakaya and Isaacson, 1981). These effects can be split mathematically in two kinds of components: first order and second order.
The instantaneous force in heave and moment in pitch are given in the form (Richards and Stoten, 1981; Dumlu and Istepanopulos, 1995; Liciega-Castro and VanDer Molken, 1995):

\[ Z_{\text{wave}}(t) = Z_0(t) + Z_2(t) \]

\[ Z_i(t) = \sum_{j=1}^{N} C_{ij} N \sin \left( \frac{2 \pi j}{g} \right) \frac{V_p (3 + \sin^2 \mu)}{g} \left[ \frac{\mu}{1 + \mu^2} \right] C_{22} \frac{1 - 0.04U \cos \mu}{1 - 0.02U \cos \mu} F_{s,2} \sin \omega_{ij} t \]

\[ Z_2(t) = -\sum_{i=1}^{N} \sum_{j=1}^{N} F_{s,2} \sin \omega_{ij} t \sin \omega_{ij} t \]

\[ M_{\text{wave}}(t) = M_1(t) + M_2(t) \]

\[ M_1(t) = -\sum_{i=1}^{N} C_{M1} \frac{1 - 0.02U \cos \mu}{1 - 0.02U \cos \mu} S_{\sin} \left( \omega_{ij} \right) F_{s,2} \cos \omega_{ij} t \]

\[ M_2(t) = -C_{M2} \cos \omega_{ij} t Z_2(t) \]

(3)

where \( F_{s,2} \) is the force due to the attenuated static head at the vehicle depth produced by wave components:

\[ F_{s,2} = a_i \omega_i^2 e^{-\frac{\alpha_i^2 \mu}{g}} \]

\[ a_i = \sqrt{2S(\omega_i) \delta \omega} \]

and:

\[ \nabla - \text{submarine volumetric displacement}, \]

\[ C_{Z1}, C_{Z2}, C_{M1}, C_{M2} - \text{hydrodynamic coefficients}, \]

\[ \mu - \text{heading of the submarine relative to the waves.} \]

In addition, the forces on the submarine are dependent on the encounter frequency \( \omega_{0c} \) rather than on the sea state alone, where:

\[ \omega_{ei} = \omega_i - \frac{\omega_i^2 U}{g} \cos(\mu) \]

Spectral factorisation techniques are used to express \( Z_{\text{wave}} \) and \( M_{\text{wave}} \) in a more suitable form for the use in designing an optimal stochastic control. The sea spectrum was modelled by transfer function representation (colour filter). This model is written in the form of two cascaded second-order sections. In the published submarine periscope depth control research, these disturbances were modelled by the simple affine form:

\[ Z_{\text{wave}}(t) \approx a \nu(t) + b \]

\[ M_{\text{wave}}(t) \approx c \nu(t) + d \]

(5)

where \( \nu(t) \) is instantaneous sea surface wave height, and constants \( a, b, c, d \) are determined by data fitting in (3,4). The main disadvantage of this approach is a poor correspondence with the real situation (Mandžuka, 1998b). The dynamic model is inadequate, and there is no phase information between \( Z_{\text{wave}}(t), M_{\text{wave}}(t) \) and \( \nu(t) \).

The well-known Pierson-Moskowitz (PM) sea-surface power spectral density function is given by the following non-rational expression:

\[ S_{PL}(\omega) = A \frac{\exp(-\frac{B}{\omega^2})}{\omega^5} \]

where the values for \( A \) and \( B \) were suggested by the ITTC (International Towing Tank Conference) as:

\[ A = \alpha \frac{g^2}{2}, \]

\[ \alpha - \text{Phillips constant, } (\alpha=0.0081), \]

\[ B = 3.109h_s^{-2}, \]

\[ h_s - \text{significant wave height.} \]

The depth extended of PM model is the basic idea for the new approach. The sea wave is attenuated at depth \( H \) by the factor:

\[ K(H) = e^{-\frac{\alpha_i^2 U}{g}} \]

(6)

The power spectra for displacement, velocity and acceleration may be derived as:

\[ S_{k}^{PM}(H,\omega) = \omega^2 e^{-\frac{2H}{g}} S_{PM}(\omega) \]

where:

\[ k=0 \text{ for displacement, } k=1 \text{ for velocity, } k=2 \text{ for acceleration.} \]

The description of the attenuation factor by the first order lag is a reasonable choice. The new transfer function of sub-surface wave dynamics is presented in the form:

\[ G_{H}(s) = \frac{1}{1 + T_H s} \]

(7)

where \( G_{PM}(s) \) is sea-surface transfer function model (Mandžuka, 1994). The first order approximation for \( T_H \) is given by the Taylor series expansion of the spectral representation of attenuated factor \( K(H) \), and it is given as:

\[ T_{H0} = \sqrt{\frac{2H}{g}} \]

(8)

This approximation is overestimated and its accuracy decreases with depth. A better solution of the above problem may be given in the form of minimising the integral squared error cost function:

\[ J_0(T_H) = \frac{1}{2} \sum_{\omega_1} \left( S_{H}(\omega) - \|G_{H}(s)\|^2 \right)^2 d\omega \]

(9)

where \( \omega_1 \) and \( \omega_2 \) define the frequency interval (window) of interest, \( S_{H}(\omega) \) original sea spectrum deep-attenuated, and \( G_{H}(s) \) is the wanted approximation in the form (7). The estimated first order approximation (8) is a good starting value. The results for a few significant wave heights \( h_s \) and sea depths \( H \) are shown in Table 1.
Table 1. The values of $T_H$

<table>
<thead>
<tr>
<th>H[m]</th>
<th>$T_{H\mu}$[s]</th>
<th>$h_s=0.5$</th>
<th>$h_s=1$</th>
<th>$h_s=1.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0</td>
<td>1.0096</td>
<td>2.3615</td>
<td>1.6199</td>
<td>1.4174</td>
</tr>
<tr>
<td>7.5</td>
<td>1.2365</td>
<td>4.1283</td>
<td>2.4035</td>
<td>1.9840</td>
</tr>
<tr>
<td>10.</td>
<td>1.4278</td>
<td>6.7044</td>
<td>3.3396</td>
<td>2.6057</td>
</tr>
</tbody>
</table>

The graph presentation of the original sub-surface Pierson-Moskowitz wave spectrum (solid line) and its approximation (dotted line) is given in Fig. 2.

Fig.2 The sub-surface Pierson-Moskowitz spectrum and its approximation

The new linear submarine wave disturbance model (oscillatory components) can be presented in the following form:

$$
\begin{align*}
\dot{x}_s(t) &= A_s x_s(t) + G_s w(t) \\
y_s(t) &= C_s x_s(t) \\
y_s(t) &= \left[v_H \cdot f'_w \cdot f'_q\right]^T
\end{align*}
$$

where:

$$
A_s = \begin{bmatrix} 1/T_H & 0 & 0 & 0 & 0 \\
-2c_{q} & -\alpha_0^2 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & -2c_{q} & -\alpha_0^2 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix},
\quad G_s = \begin{bmatrix} 1/T_H & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

The coefficients $c_{qw}$ and $c_{q\theta}$ are dependent on the submarine velocity and the encounter angle. In the above mathematical model (10), the pitch disturbance is approximated around resonant frequency $\omega_0$. The facts listed below are the reason for this approximation:

-the main energetic part of sea spectrum lies near the modal (resonant) frequency,

-there is adequate phase information between heave and pitch disturbances,

-the order of mathematical model is unchanged.

The terms in matrices $G$ and $H$, are defined as:

$$
g_{11} = \frac{C_{Z_1}}{1 - Z_{\omega}}, \quad g_{22} = \frac{C_{M_1}}{1 - M_{\dot{\theta}} r_{\theta}^2} $$

$$
h_{11} = \frac{C_{Z_2}}{1 - Z_{\omega}}, \quad h_{22} = \frac{C_{M_2} C_{s2}}{1 - M_{\dot{\theta}} r_{\theta}^2}
$$

where:

$C_{Z1}, C_{M1}, C_{Z2}, C_{M2}$ - hydrodynamic coefficients, $r_{\theta}$ - radius of gyration.

The drift components are calculated as:

$$
s_w = -c_d d \\
s_q = -c_d d \\
c_s = \frac{3 + \sin^2 \mu}{g \cdot 10^{\sin \mu}} (1 - 0.04 U \cos \mu)
$$

where the constant term $d$ is given by the approximation method based on the linearisation technique. The simplest form of linearisation is based upon the expansion of the nonlinear function in a Taylor series about some operating point, retaining only linear terms in the analysis (Billings, 1985). The constant term $d$ (for square nonlinearity) is given as:

$$
d = M_{2s}^{PM}(H)
$$

where $M_{2s}^{PM}(H)$ is the sub-surface second spectral moment at a depth $H$. The procedure for the sub-surface spectral moment calculation is given in (Gran, 1992; Mandzuka, 1998a). The simulation results of the submarine HF-disturbances are shown in Fig. 3. It can be noticed that the pitch signal is smoother than the heave signal.

High-frequency impulse responses are presented in Fig. 4. It is obvious that there is a good phase matching with equations (3) and (4). The drift components $s_w$ and $s_q$ are included in the submarine mathematical model as constant inputs at a depth $H$. There are some reasons for this approach. First, the basic high-frequency model is unchanged, and second this approach is suitable for a lot of modern control design techniques (min-max, $H_{\infty}$). Particularly, it is possible to use some of feedforward techniques (Booth, 1983).
various types of inclination sensors. The characteristics of a sensor are generally described in the form of its lag and the measurement noise. The lag of submarine sensors is negligible. The measurement noise is described as Gaussian white noise with zero mean and corresponding covariance matrix.

The compact mathematical model of the submarine dynamics may be presented in the following form:

\[
\begin{aligned}
\dot{x}_T(t) &= A_T x_T(t) + B_T u(t) + G_T w(t) + H_T \eta(t) \\
x_T(t) &= [x_v(t), x_s(t)]^T \\
y_T(t) &= C_T x_T(t) + \eta(t)
\end{aligned}
\]  

(13)

where:

\[
A_T = \begin{bmatrix} A_v & 0 \\ 0,5,4 \end{bmatrix}, \quad B_T = \begin{bmatrix} B_v \\ 0,5,2 \end{bmatrix}, \quad G_T = \begin{bmatrix} 0,5,1 \\ [G_v] \end{bmatrix}
\]

\[
H_T = \begin{bmatrix} H_v \\ 0,5,4 \end{bmatrix}, \quad C_T = \begin{bmatrix} C_v \\ 0,5,1 \\ 0,5,3 \end{bmatrix}
\]

Particular measurement matrices are presented in the usual (MATLAB) form:

\[
C_v = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

\[
C_{ss} = C_s(1,:) \\
C_{sd} = C_s(2:3,:)
\]

The suitable process and measurement covariance matrices are given in the form:

\[
Q_w = E[w(t)w^T(t)] = 1.
\]

\[
R_\eta = E[\eta(t)\eta^T(t)] = \begin{bmatrix} \sigma_\eta^2 & 0 \\ 0 & \sigma_\theta^2 \end{bmatrix}
\]

where:

\[
\sigma_\eta^2 - \text{variance of depth sensor measurement noise},
\sigma_\theta^2 - \text{variance of pitch sensor measurement noise}.
\]

4. SUBMARINE COMPACT MODEL

Based on the principle of superposition it is possible to represent the submarine dynamics as the sum of low- and high-frequency motions. The next property of the submarine control system is the application of a pressure type sensor for depth measurement. The total pressure is comprised of hydrostatic head of water from the mean sea level plus a head which is a function of continuously changing instantaneous wave height at sea surface.

\[
y_{ps}(t) = H + K(H) * v(t)
\]

(12)

where attenuation factor K(H) is given by (6). This property is sometimes referred to as wave noise and it is a serious nuisance (Booth, 1983). There are two possible solutions to avoid this problem; first, to use an inertial depth sensor, and second to use some combinations of sensors and modern signal processing. The correct mathematical model of the submarine dynamics is very important for the last approach. The pitch measurement is realised by
frequency wave forces. For a deeply submerged submarine, small motions are analysed using the concept of hydrodynamic coefficients. These represent a Taylor series expansion of the functional relationship between body movements and the resulting fluid forces. The effects of the surface waves on a submarine at the periscope depth is presented in the form of a transfer function. The principle of superposition is used to derive the submarine disturbance forces model. The basic conclusions of the paper are given below:

1. Using the proposed mathematical model of a submarine at the periscope depth a lot of disadvantages of the former research are avoided. In the proposed mathematical model there is better phase information between the main components of wave disturbances.

2. As additional states were added to the new mathematical model, the level achieved by modern control law optimisations is generally improved.

3. The advanced signal processing technique may be used with the new submarine mathematical model.

The requirements for submarine periscope depth operations have been increased by integration with carrier battle groups, littoral operations, laying seabed mines etc.. Operating at the periscope depth exhausts the submarine control at the surface. To counter these forces, the submarine's ballast may be changed and/or control surfaces are used. Because of the stochastic nature of waves, the manual operation of the submarine for longer periods at the periscope depth exhausts the submarine control operators. Improved periscope depth performance is therefore imperative (Papoulias, 1995).

REFERENCES


